1 Chapter 8: Comparative Static Analysis of General Function Models

1. General Form

National Income model

1.0.1 Specific

(1) : \( Y = C + I_0 + G_0 \)

(2) : \( C = a + b(Y = T_0) \)

By substitution

\[
Y = a - b(Y - T_0) + I_0 + G_0
\]

Solution

\[
Y^e = \frac{a + I_0 + G_0 - bT_0}{1 - b}
\]
1.0.2 General

\[
Y = Y(C, I_0, G_0) \\
C = C(Y, T_0) \\
Y = Y(C(Y, T_0), I_0, G_0) \\
Y^e = Y^e(I_0, G_0, T_0)
\]

The general form can be expressed as:

\[
Y^e = C(Y^e, T_0) + I_0 + G_0
\]

\(\frac{\partial Y^e}{\partial T_0}\) has a direct and indirect effect:

\[
\frac{\partial C}{\partial T_0} \text{ and } \frac{\partial C \partial Y^e}{\partial Y^e \partial T_0}
\]

1.1 Differentials

Given \(y = f(x)\)

Then \(\frac{dy}{dx} = f'(x)\)

But also \(\frac{dy}{\text{Change in Y}} = f'(x) \frac{dx}{\text{A converter Change in X}}\)

\(f'(x)\) "converts" a \(\Delta\) in \(x\) into a \(\Delta\) in \(Y\)

Example:

\[y = x^2 \Rightarrow dy = 2xdx\]

at \(x = 2; y = 4\), if \(dx = .01\) then \(dy = 2(2)(0.01) = 0.4\)

Therefore: as \(x\) \(\Delta's\) from 2 to 2.01 then \(y\) \(\Delta's\) from 4 to 4.04
1.1.1 Differentials and Point Elasticity

From ECON 200

Are Elasticity $= \frac{\Delta Q}{\Delta P}$ or $\frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$

Point Elasticity

$\epsilon^d = \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{\text{Marginal}}{\text{Average}}$

Example

Let $Q = a - bP$

Then $\frac{dQ}{dP} = -b$ and $\frac{P}{Q} = \frac{P}{a - bP}$

Therefore

$\epsilon^d = \frac{-bP}{a - bP}$

Let $Q = 10 - 2P$

Then $\epsilon^d = \frac{-2P}{10 - 2P}$

$\epsilon^d = \frac{-P}{5 - P}$

1.2 Total Differentials

Consider the Utility Function:

$U = U(x, y)$

Totally differentiate

$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$

or $dU = MU_x dx + MU_y dy$
Indifference Curve: \[ dU = 0 \]
\[ MU_x dx + MU_y dy = 0 \]
\[ MU_y dy = -MU_x dx \]
\[ dy = \frac{-MU_x}{MU_y} dx \text{ (iff } MU_y \neq 0!!) \]
\[ \frac{dy}{dx} = \frac{-MU_x}{MU_y} = MRS \]

Graphically

Note: if \( dx \neq 0 \) then \( dy = 0 \) but both \( MU_x, MU_y \neq 0 \) (from Economic Theory). Therefore minus sign (-) in front of \(-\frac{MU_x}{MU_y}\)

Example

1.

if \( U(x,y) = xy \)
then \( dU = ydx + xdy \)
and \( MRS = \frac{dy}{dx} = -\frac{y}{x} \)

2.

if \( U(x,y) = x^2y^2 \)
then \( dU = 2xy^2dx + 2x^2ydy \)
and \( MRS = \frac{dy}{dx} = -\frac{2xy^2}{2x^2y} = -\frac{y}{x} \)

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\(^1\)Graph page 5 chapter 8
3.

\[ \text{if } U(x, y) = x + y \]
\[ \text{then } dU = dx + dy \]
\[ \text{and } MRS = \frac{dy}{dx} = -1 \]

1.2.1 Total Differentials: Generally

Let \( U = U(x_1, x_2, \ldots x_n) \)

Then \( dU = \sum \frac{\partial U}{\partial x_i} dx_i \) + \( \sum \frac{\partial U}{\partial x_j} dx_j \) + \( \ldots \) + \( \frac{\partial U}{\partial x_n} dx_n \)

Or \( dU = U_1 dx_1 + U_2 dx_2 + \ldots U_n dx_n \)

where \( U_i = \frac{\partial U}{\partial x_i} \) (the partial derivative)

If \( dx_2 = dx_3 = \ldots dx_n = 0 \)

Then \( dU = \frac{\partial U}{\partial x_1} dx_1 + (0) \)

Then \( \frac{dU}{dx_1} = \frac{\partial U}{\partial x_1} = U_1 \)

*The partial derivative of a function is simply the total differential with all but one of the \( dx_i \)'s set equal to zero.

1.2.2 Rules of Differentials

1. \( dk = 0 \)

2. \( y = ax^n \Rightarrow dy = \frac{\partial U}{\partial x} \cdot dx = anx^{n-1}dx \)

3. \( y = x_1 + x_2 \Rightarrow dy = dx_1 + dx_2 \)
4. $y = x_1 x_2 \Rightarrow dy = x_2 dx_1 + x_1 dx_2$

5. $y = \frac{x_1}{x_2} \Rightarrow dy = \frac{x_2 dx_1 - x_1 dx_2}{x_2^2}$

Example

\[
\begin{align*}
y &= x^3_1 + 3x^2_2 + 4x_1x_2 \\
dy &= \frac{dy}{dx_1} dx_1 + \frac{dy}{dx_2} dx_2 \\
\frac{\partial y}{\partial x_1} &= 3x^2_1 + 4x^2 \\
\frac{\partial y}{\partial x_2} &= 6x_2 + 4x_1 \\
dy &= (3x^2_1 + 4x^2) \, dx_1 + (6x_2 + 4x_1) \, dx_2
\end{align*}
\]
Further Examples

\[ y = \frac{(x_1 + x_2)^2}{x_2^3} \]

\[ dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \]

\[ \frac{\partial y}{\partial x_1} = \left( \frac{1}{x_2^3} \right) 2 (x_1 + x_2) = \frac{2 (x_1 + x_2)}{x_2^3} \]

\[ \frac{\partial y}{\partial x_2} = \frac{[x_2^3 (x_1 + x_2) (2)] - [(x_1 + x_2)^2 (3)(x_2^3)]}{(x_2^3)^2} \]

\[ \frac{\partial y}{\partial x_2} = \frac{2x_2^3 (x_1 + x_2) - 3 (x_1 + x_2)^2 x_2}{x_2^6} \]

\[ \frac{\partial y}{\partial x_2} = \frac{2x_2 (x_1 + x_2) - 3 (x_1 + x_2)^2}{x_2^4} \]

\[ dy = \left[ \frac{2 (x_1 + x_2)}{x_2^3} \right] dx_1 + \left[ \frac{2x_2 (x_1 + x_2) - 3 (x_1 + x_2)^2}{x_2^4} \right] dx_2 \]
1.2.3 Cobb-Douglas Production Function

\[ Q = Q(K,L) = K^aL^b \]

\[ dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL = MP_K dK + MP_L dL \]

\[ \frac{\partial Q}{\partial K} = [aK^{a-1}L^b] = a \left( \frac{K^aL^b}{K} \right) = a \frac{Q}{K} \]

\[ \frac{\partial Q}{\partial L} = [bK^aL^{b-1}] = b \left( \frac{K^aL^b}{L} \right) = b \frac{Q}{L} \]

\[ dQ = \left[ a \frac{K^aL^b}{K} \right] dK + \left[ b \frac{K^aL^b}{L} \right] dL \]

\[ dQ = \left[ a \frac{dK}{K} + b \frac{dL}{L} \right] \cdot K^aL^b \]

\[ dQ = \left[ a \frac{dK}{K} + b \frac{dL}{L} \right] \cdot Q \]

\[ \frac{dQ}{Q} = (a + b) \frac{dS}{S} = \frac{dQ}{S} \frac{S}{dS} = (a + b) \quad \text{Elasticity of Scale} \]

1.3 Total Derivatives and the Chain Rule

Let \( y = y(x,z) \) and \( x = x(z) \)

\[ dy = \left( \frac{\partial y}{\partial x} \right) dx + \left( \frac{\partial y}{\partial z} \right) dz \quad \text{and} \quad dx = \frac{dx}{dz} \frac{dz}{dz} \]

Substitute \( dy = \left( \frac{\partial y}{\partial x} \right) \left( \frac{dx}{dz} \right) dz + \left( \frac{\partial y}{\partial z} \right) dz \)

Divide by \( dz \)

\[ \frac{dy}{dz} = \left( \frac{\partial y}{\partial x} \right) \frac{dx}{dz} + \left( \frac{\partial y}{\partial z} \right) \frac{dz}{dz} \quad \left\{ \frac{dz}{dz} = 1 \right\} \]
The Total Derivative $\frac{dy}{dz}$ is:

\[
\frac{dy}{dz} = \left( \frac{\partial y}{\partial x} \right) \left( \frac{dx}{dz} \right) + \left( \frac{\partial y}{\partial z} \right)
\]

Total $\Delta$ in $Y$ from $\Delta$ in $z$  The indirect effect of $z$ on $y$ through $x$  The direct effect of $z$ on $y$

1.3.1 Chain Rule

\[
y = y(x, z) \text{ but } x = x(z)
\]

Therefore $y = y(x(z), z) \{y = f(z)\}$

$y$ is a function of one exogenous variable

\[
\frac{dy}{dz} = \frac{\partial y}{\partial x} \frac{dx}{dz} + \frac{\partial y}{\partial z}
\]

$\text{Indirect}$
$y \leftarrow x \leftarrow z$

$\text{Direct (z to y)}$
Example

\[ y = (x + 2)^2 + zx + z^2 \]
\[ x = 2z + 3 \]
\[ \frac{dy}{dz} = \frac{\partial y}{\partial x} \frac{dx}{dz} + \frac{\partial y}{\partial z} \frac{dz}{dz} \]
\[ (1) \frac{\partial y}{\partial x} = [2(x + 2) + z] \quad (2) \frac{\partial y}{\partial z} = [x + 2z] \quad (3) \frac{dx}{dz} = 2 \]

\[ \frac{dy}{dz} = (2x + 4 + 2) (2) + (x + 2z) \]

sub in \( x = (2z + 3) \)
\[ \frac{dy}{dz} = (2(2z + 3) + 4 + 2) (2) + ((2z + 3) + 2z) \]
\[ \frac{dy}{dz} = (10z + 20) + (4z + 3) = 14z + 23 \]

Alternative Method

\[ y = (x + 2)^2 + zx + z^2 \]
\[ x = 2z + 3 \]
\[ y = ((2z + 3) + 2)^2 + z(2z + 3) + z^2 \]
\[ y = (2z + 5)^2 + 3z^2 + 3z \]
\[ \frac{dy}{dz} = 2(2z + 5)(2) + 6z + 3 \]
\[ \frac{dy}{dz} = 8z + 20 + 6z + +3 = 14z + 23 \]

2 approaches for \( y = y(x, z) \) and \( x = x(z) \)
1.  

\[ dy = \frac{\partial y}{\partial x} \, dx + \frac{\partial y}{\partial z} \, dz \]

sub in for \( dx = \frac{dx}{dz} \, dz \)

\[ dy = \left[ \frac{\partial y}{\partial x} \frac{dx}{dz} + \frac{\partial y}{\partial z} \right] \, dz \]

2.  

sub \( x(z) \) into \( y(x, z) \)

\[ y = y(x(z), z) \]

\[ y = g(z) \] "g" is a new function

\[ \frac{dy}{dz} = g'(z) \]

Further Examples

\[ y = y(x_1, x_2, \alpha, \beta) \]

and \( x_1 = x_1(\alpha, \beta) \) \( x_2 = x_2(\alpha, \beta) \)

\[ dy = \left[ \left( \frac{\partial y}{\partial x_1} \right) \left( \frac{dx_1}{d\alpha} \right) + \left( \frac{\partial y}{\partial x_2} \right) \left( \frac{dx_2}{d\alpha} \right) + \frac{\partial y}{\partial \alpha} \right] \, d\alpha \]

\[ + \left[ \left( \frac{\partial y}{\partial x_1} \right) \left( \frac{dx_1}{d\beta} \right) + \left( \frac{\partial y}{\partial x_2} \right) \left( \frac{dx_2}{d\beta} \right) + \frac{\partial y}{\partial \beta} \right] \, d\beta \]

\( y \) is a function of 4 variables but only 2 exogenous variables \( (\alpha, \beta) \)

Find \( \frac{dy}{d\alpha} \), (the total derivative w.r.t. \( \alpha \))

1. set \( d\beta = 0 \) (the second term drops out)

2. divide by \( d\alpha \)

\[ \frac{dy}{d\alpha} = \left[ \left( \frac{\partial y}{\partial x_1} \right) \left( \frac{dx_1}{d\alpha} \right) + \left( \frac{\partial y}{\partial x_2} \right) \left( \frac{dx_2}{d\alpha} \right) + \frac{\partial y}{\partial \alpha} \right] \frac{d\alpha}{d\alpha} \quad (\frac{d\alpha}{d\alpha} = 1) \]
1.3.2 Differentials and Derivatives

\[ y = y(x) \]
\[ dy = y'(x)dx \]

or \[ dy = \frac{dy}{dx}dx \]

Divide both sides by \( dx \)

\[ \frac{dy}{dx} = \frac{dy}{dx} \]

LHS: is a ratio of two differentials
RHS: is NOT a ratio of two differentials.

RHS is the derivative \( \frac{dy}{dx} = y'(x) \)

1.4 Implicit Functions

Explicit Function

\[ y = f(x) \]

Rewritten as an Implicit Function

\[ y - f(x) = 0 \]

In General:

\[ F(y, x) = 0 \]
\[ F(y, x) = k \text{ (where } k \text{ is some constant or parameter} \]

Any explicit function, \( y=f(x) \), can be expressed as an implicit function, \( F(y,x)=0 \), however, not all implicit functions can be expressed as explicit functions directly.
An implicit function: \( F(y, x_1, \ldots, x_n) = 0 \) may define \( y \) as a function of \( x_1, \ldots, x_n \), yet cannot be solved directly for \( y = f(x_1, \ldots, x_n) \) (this may hold only over a limited range of \( F \), but not everywhere).

We can tell if \( F(y, x_1, \ldots, x_n) \) does indeed implicitly define \( y \) as a function of \( x_1, \ldots, x_n \) by use of the IMPLICIT FUNCTION THEOREM.

**THEOREM:**

1. (a) if \( F \) has continuous partial derivatives \( F_y, F_1, F_2, \ldots, F_n \) and
2. (b) at the point we are interested in \( F_y \neq 0 \) at \( y = y_0 \)

Then at \( y = y_0 \) \( F \) implicitly defines \( y \) as a function of \( x_1, \ldots, x_n \).(at some value \( y = y_0 \) \( F=0 \) is an identity)

Suppose:

\[
F(y, x_1, x_2) = 0
\]

(if the values of \( y, x_1, x_2 \) are the onesthat satisfy this equation, then this equation is an identity)

However, this function cannot be solved explicitly for

\[
y = f(x_1, x_2)
\]

We can still find

\[
\frac{\partial y}{\partial x_1} \text{ and } \frac{\partial y}{\partial x_2}
\]

Through the use of Total Differentials

\[
dF = F_y dy + F_1 dx_1 + F_2 dx_2 = 0
\]

Let \( dx_2 = 0 \)

Then

\[
F_y dy + F_1 dx_1 = 0
\]

\[
F_y dy = -F_1 dx_1
\]

\[
\frac{\partial y}{\partial x} = \frac{dy}{dx_1} \bigg|_{dx_2=0} = \frac{-F_1}{F_y} \quad \{F_y \neq 0\}
\]
1.4.1 Implicit Function Rule

Given:

\[ F(y, x_1, \ldots x_n) = 0 \]

Then:

\[ \frac{\partial y}{\partial x_i} = \frac{-F_i}{F_y} \]

The partial derivative is interpreted as a ratio of two differentials.

Example:

\[ \tilde{U} = U(y, x) = x^{1/2}y^{1/2} \]

For \( dU = 0 \)

\[ \frac{dy}{dx} = \frac{-U_x}{U_y} = -\left(\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}\right) = -\frac{y}{x} = MRS \]

Explicitly:

\[ y = \frac{\tilde{U}^2}{x} \{ \tilde{U}^2 = \text{constant} \} \]

\[ \frac{\partial y}{\partial x} = -\frac{\tilde{U}^2}{x^2} = -\left(\frac{\tilde{U}^2}{x}\right)\frac{1}{x} = -\frac{y}{x} \]

Or:

\[ \frac{\partial F}{\partial y_1} dy_1 + \frac{\partial F}{\partial y_2} dy_2 = \left( -\frac{\partial F}{\partial x_1} dx_1 \right) + \left( -\frac{\partial F}{\partial x_2} dx_2 \right) \]

\[ \frac{\partial G}{\partial y_1} dy_1 + \frac{\partial G}{\partial y_2} dy_2 = \left( -\frac{\partial G}{\partial x_1} dx_1 \right) + \left( -\frac{\partial G}{\partial x_2} dx_2 \right) \]
In Matrix Form:

\[
\begin{bmatrix}
\frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} \\
\frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2}
\end{bmatrix}
\begin{bmatrix}
dy_1 \\
dy_2
\end{bmatrix}
= 
\begin{bmatrix}
-\frac{\partial F}{\partial x_1}dx_1 - \frac{\partial F}{\partial x_2}dx_2 \\
-\frac{\partial G}{\partial x_1}dx_1 - \frac{\partial G}{\partial x_2}dx_2
\end{bmatrix}
\]

"Jacobian"

Test for exisitance by the Determinant

\[
|J| = \left(\frac{\partial F}{\partial y_1}\right)\left(\frac{\partial G}{\partial y_2}\right) - \left(\frac{\partial F}{\partial y_2}\right)\left(\frac{\partial G}{\partial y_1}\right) \neq 0
\]

If \( |J| = 0 \) then \( y_1 \) and \( y_2 \) are not functions of \( x_1 \) and \( x_2 \)
\( |J| = 0 \) is the same as \( f_y \neq \) in single equation case.

Jacobian: Matrix of "Partial Derivatives" with respect ot the "Endogenous variables" where the partial derivative and are treated as constants.