1 Simple Compound Interest

Suppose you have \( x \) dollars to invest at an interest rate of \( r \) percent per year. In one year you will have \( y \) dollars, where

\[
y = x + rx = x(1 + r)
\]

in two years

\[
y = [x(1 + r)](1 + r) = x(1 + r)^2
\]

in three years

\[
y = [x(1 + r)^2](1 + r) = x(1 + r)^3
\]

The present value (PV) of \( y \) 3 years from now is

\[
x = \frac{y}{(1 + r)^3} = y(1 + r)^{-3}
\]

PV: Tells you "how much to invest now" in order to have \( y \) dollars in 3 years.
1.1 Compounding Within a Year

1. (a) Semi-annual compounding
   at six months
   \[ y = x \left(1 + \frac{r}{2}\right) = x + \frac{xr}{2} \]
   at one year
   \[ y = \left[x \left(1 + \frac{r}{2}\right)\right] \left(1 + \frac{r}{2}\right) = x \left(1 + \frac{r}{2}\right)^2 \]

   (b) Monthly compounding
   \[ y = x \left(1 + \frac{r}{12}\right)^{12} \text{ for one year} \]
   \[ y = \left[x \left(1 + \frac{r}{12}\right)^{12}\right] \left(1 + \frac{r}{2}\right) = x \left(1 + \frac{r}{12}\right)^{24} \text{ for two years} \]
   \[ y = x \left(1 + \frac{r}{12}\right)^{12n} \text{ for } n \text{ years} \]

1.2 Converting Compound Interest into an Annual Yield

Suppose you are offered a choice:

1. 10% compounded semi-annually, or
2. 10.2% annually

Which would you choose?

We know for semi-annual
\[ y = x \left(1 + \frac{r}{2}\right)^2 = x \left(1 + \frac{10}{2}\right)^2 = (1.05)^2 x \]
\[ y = 1.1025x \]
Yield = $y - \text{principal} = y - x$

Yield = 1.1025$x - x = 0.1025x$ or you can earn 10.25% annually since 10.25% > 10.20% $\implies$ Pick option (1)

### 1.3 Continuous Compounding

1. (a) Daily interest for one year

\[ y = x \left(1 + \frac{r}{365}\right)^{365} \]

Suppose $x=$$1 and $r=100\%$ (or $r=1$)

\[ y = 1 \left(1 + \frac{1}{365}\right)^{365} = 2.71456 \]

(b) Compound hourly ($365 \times 24=8760$)

\[ y = x \left(1 + \frac{1}{8760}\right)^{8760} = 2.71812 \]

or if

\[ y = 1 \left(1 + \frac{1}{m}\right)^{m} \]

if we let $m \implies$ infinity ($\infty$)

\[ y = \left(1 + \frac{1}{m}\right)^{m} \implies 2.71828\ldots \equiv e \]

for any $r$ as $m \rightarrow \infty \{\text{and } x = \$1\}$

\[ y = \left(1 + \frac{1}{m}\right)^{m} \implies e^{r} \]
2 The Number "e"

The number $e = 2.71828...$ is the value of $1$ compounded continuously for one year (or one period) at an interest rate of 100%.

Continuous compounding at $r$ percent for $t$ years of a principal equal to $x$

$$y = xe^{rt}$$

The present value of $y$ is

$$x = \frac{y}{e^{rt}} = ye^{-rt}$$

which tells you the amount needed to invest today that will be worth $y$ dollars in $t$ years of continuous compounding

Present Value ($xe^{rt}$) Graphically

Slope $= \frac{dy}{dt} = rxe^{rt}$

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3 Derivative rules of $e$

1. 
   
   \[ y = e^x \quad \frac{dy}{dx} = e^x \]

2. 
   
   \[ y = e^{f(x)} \quad \frac{dy}{dx} = f'(x)e^{f(x)} \]

3. Examples:
   
   (a) \( y = e^{3x} \) \quad \frac{dy}{dx} = 3e^{3x} 
   
   (b) \( y = e^{-rt} \) \quad \frac{dy}{dt} = -re^{-rt} 
   
   (c) \( y = ae^{(t^2-t)} \) \quad \frac{dy}{dt} = a(2t-1)e^{(t^2-t)} 

4. 
   
   \[ e^{-\infty} = \frac{1}{e^{\infty}} \approx 0 \quad e^0 = 1 \]

3.1 Growth Rates

Given 

\[ y = xe^{rt} \]

The change in \( y \) is 

\[ \frac{dy}{dt} = rxe^{rt} = ry \]

However, the percentage change in \( y \), or the "growth rate" is 

\[ \text{Growth Rate} = \frac{\Delta \ln y}{y} \approx \frac{dy}{y} \]

Therefore

\[ \text{Growth Rate} = \frac{\frac{dy}{dt}}{y} = \frac{rxe^{rt}}{xe^{rt}} = r \]
Where \( r \) is the continuous rate of growth of \( y \) over time. NOTE: the growth rate is constant, however, the slope of \( y = xe^{rt} \) is not constant.

4 Logarithms

4.1 Common Log (or log base 10)

Given

\[ 10^2 = 100 \]

The exponent 2 is defined as the logarithm of 100 to the base 10.

eg.

\[ \log 1000 = 3 \quad \text{because} \quad \{10^3 = 1000\} \]
\[ \log 10 = 1 \quad \text{because} \quad 10^1 = 10 \]
\[ \log 1 = 0 \quad \text{because} \quad 10^0 = 1 \]
\[ \log 0.1 = -1 \quad \text{because} \quad 10^{-1} = .1 \]
\[ \log 0.01 = -2 \quad \text{because} \quad 10^{-2} = .001 \]

4.2 Natural Logarithm

If \( y = e^x \) \( \ln y = \ln e^x = x \) where \( ln \) is the logarithm to base \( e \)

4.3 Rules of Logarithms

1. \( \ln(AB) = \ln A + \ln B \)
2. \( \ln \left(\frac{A}{B}\right) = \ln A - \ln B \)
3. \( \ln(A^b) = b \ln A \)
4.3.1 Example:
\[ \ln(x^3y^2) = 3\ln x + 2\ln y \]

4.3.2 Other Properties

if \( x = y \) then \( \ln x = \ln y \)
if \( x > y \) then \( \ln x > \ln y \)

**\( \ln(-3) \) does NOT exist!! You cannot take a logarithm of a negative number.
**\( \ln(A + B) \neq \ln A + \ln B \)!!

5 Derivatives of the Natural Logarithm

1. \( y = \ln x \quad \frac{dy}{dx} = \frac{1}{x} \) or \( dy = \frac{dx}{x} \)

2. \( y = \ln ax \quad \frac{dy}{dx} = \frac{a}{ax} = \frac{1}{x} \)

   OR \( y = \ln ax = \ln x + \ln a \)

   \[ \frac{dy}{dx} = \frac{1}{x} \left\{ \text{since} \quad \frac{d(\ln a)}{dx} = 0 \right\} \]

3. \( y = \ln(x^2 + 2x) \)

   \[ \frac{dy}{dx} = \frac{1}{x^2+2x} (2x + 2) = \frac{2x+2}{x^2+2x} = \frac{1}{x+2} + \frac{1}{x} \]

   OR \( y = \ln(x^2 + 2x) = \ln [(x + 2) x] = \ln(x + 2) + \ln x \)

   \[ \frac{dy}{dx} = \frac{1}{x+2} + \frac{1}{x} \]
6 Optimal Timing Problems

6.1 The Forest Harvesting Problem

Assume a stand of trees grows according to the following function

\[ V(t) = Ae^{\alpha - \beta t} \quad \{ \text{measured in (ft)}^3 \} \]

Question: When is the best time to harvest the stand of trees?

· For simplicity, assume that the price per ft$^3$ for lumber is $1 and it remains constant over time

· if the market rate of interest is $r$ then the problem is to choose a time to harvest the trees that maximizes the present value of the asset
at any time, $t_0$ the present value is:

$$PV = V(t_0) e^{-rt_0}$$
$$= \left( A e^{\alpha-\frac{\beta}{t_0}} \right) e^{-rt}$$
$$PV = A e^{\alpha-\frac{\beta}{\tau}-rt}$$
Optimal harvest time: $t_3$

Maximum present value: $PV_3 \left\{ \frac{V'_3}{V_3} = r \right\}$

At $V_1$ the growth rate of trees exceeds the growth rate of a financial asset since $\left( \frac{V_1(t)'}{V_1} > r \right)$

Present Value is

$$PV(t) = V(t)e^{-rt}$$

$$PV(t) = Ae^{\alpha - \frac{\beta}{t} - rt} \quad \left\{ V(t) = Ae^{\alpha - \frac{\beta}{t}} \right\}$$

Max PV with respect to $t \left\{ \frac{dPV}{dt} = 0 \right\}$

$$\frac{dPV}{dt} = Ae^{\alpha - \frac{\beta}{t} - rt} \left( \frac{\beta}{t^2} - r \right) = 0$$

$$\frac{dPV}{dt} = 0 \quad \text{If} \quad \frac{\beta}{t^2} - r = 0$$

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Therefore
\[ \frac{\beta}{t^2} = r \text{ or } t = \sqrt{\frac{\beta}{r}} \]

**Logarithmic Approach**

\[
\ln PV = \ln(e^{\alpha - \frac{\beta}{t^2} - rt}) = \alpha - \frac{\beta}{t} - rt
\]

\[
\frac{d(\ln PV)}{dt} = \frac{dPV}{dt} = \frac{\beta}{t^2} - r = 0
\]

\[
= \frac{\beta}{t^2} = r \quad \left\{ \begin{array}{l}
\frac{\beta}{t^2} = \text{growth rate of the value of uncut trees} \\
r = \text{growth rate of the optimally invested money}
\end{array} \right.
\]
\[ \beta \] = the growth in your wealth from leaving trees uncut
\[ r \] = Growth in your wealth if you cut down the trees, sell them, and put the money into a savings account paying \( e^{rt} \)

Comparative Statistics: if interest rate rises: \( r \rightarrow r' \) then the optimal cutting time falls: \( t^* \rightarrow t' \)