1 OPMT 5701 Lecture Notes

1.1 Natural Logarithm and the Exponential e

1. The Number e

$$\begin{aligned} \text{if } y &= e^x \text{ then } \frac{dy}{dx} = e^x \\ \text{if } y &= e^{f(x)} \text{ then } \frac{dy}{dx} = e^{f(x)} \cdot f'(x) \end{aligned}$$

2. Examples

1. (a)

$$\begin{aligned} y &= e^{3x} \\ \frac{dy}{dx} &= e^{3x}(3) \end{aligned}$$

(b)

$$\begin{aligned} y &= e^{7x^3} \\ \frac{dy}{dx} &= e^{7x^3}(21x^2) \end{aligned}$$

(c)

$$\begin{aligned} y &= e^{rt} \\ \frac{dy}{dt} &= re^{rt} \end{aligned}$$

2. Logarithm (Natural log) $\ln x$

(a) Rules of natural log

If	$Then$
$y = AB$	$\ln y = \ln(AB) = \ln A + \ln B$
$y = A/B$	$\ln y = \ln A - \ln B$
$y = A^b$	$\ln y = \ln(A^b) = b \ln A$

NOTE: $\ln(A + B) \neq \ln A + \ln B$ EVER!!!

(b) derivatives

IF	$THEN$
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \ln(f(x))$	$\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

(c) Examples

i.

$$\begin{aligned} y &= \ln(x^2 - 2x) \\ dy/dx &= \frac{1}{(x^2 - 2x)}(2x - 2) \end{aligned}$$

ii.

$$\begin{aligned} y &= \ln(x^{1/2}) = \frac{1}{2} \ln x \\ dy/dx &= \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) = \frac{1}{2x} \end{aligned}$$

1.2 Differentials

Given the function

$$y = f(x)$$

the derivative is

$$\frac{dy}{dx} = f'(x)$$

However, we can treat dy/dx as a fraction and factor out the dx

$$dy = f'(x)dx$$

where dy and dx are called *differentials*. If dy/dx can be interpreted as "the slope of a function", then dy is the "rise" and dx is the "run". Another way of looking at it is as follows:

- $dy =$ the change in y
- $dx =$ the change in x
- $f'(x) =$ how the change in x causes a change in y

Example 1 *if*

$$y = x^2$$

then

$$dy = 2xdx$$

Lets suppose $x = 2$ *and* $dx = 0.01$. *What is the change in* $y(dy)$*?*

$$dy = 2(2)(0.01) = 0.04$$

Therefore, at $x = 2$*, if* x *is increased by 0.01 then* y *will increase by 0.04.*

1.3 Implicit Differentiation

Suppose we have the following:

$$2y + 3x = 12$$

we can rewrite it as

$$\begin{aligned} 2y &= 12 - 3x \\ y &= 6 - \frac{3}{2}x \end{aligned}$$

Now we have $y = f(x)$ and we can take the derivative

$$\frac{dy}{dx} = -\frac{3}{2}$$

Lets consider an alternative. We know that y is a function of x or, $y = y(x)$ and the derivative of y is $\frac{dy}{dx}$. If we return to our original equation, $2y + 3x = 12$, we can differentiate it IMPLICITLY in the following manner

$$\begin{aligned} 2y + 3x &= 12 \\ 2dy + 3dx &= 0 \qquad \left(\frac{d(12)}{dx} = 0\right) \\ 2\frac{dy}{dx} + 3\frac{dx}{dx} &= 0 \\ 2\frac{dy}{dx} + 3 &= 0 \qquad \left(\frac{dx}{dx} = 1\right) \end{aligned}$$

rearrange to get $\frac{dy}{dx}$ by itself

$$\begin{aligned} 2\frac{dy}{dx} &= -3 \\ \frac{dy}{dx} &= -\frac{3}{2} \end{aligned}$$

which is what we got before!

Here is a few more examples:

1.

$$\begin{aligned} y^2 + x^2 &= 36 \\ 2ydy + 2xdx &= d(36) \\ 2y\frac{dy}{dx} + 2x\frac{dx}{dx} &= 0 \qquad \left(\text{remember } \frac{d(36)}{dx} = 0\right) \\ 2y\frac{dy}{dx} + 2x &= 0 \\ \frac{dy}{dx} &= -\frac{2x}{2y} = -\frac{x}{y} \end{aligned}$$

2.

$$\begin{aligned} 5y^3 + 4x^5 &= 250 \\ 15y^2\frac{dy}{dx} + 20x^4 &= 0 \\ 15y^2\frac{dy}{dx} + 20x^4 &= 0 \\ \frac{dy}{dx} &= -\frac{20x^4}{15y^2} = -\frac{4x^4}{3y^2} \end{aligned}$$

3.

$$\begin{aligned} y^{1/2} - 2x^2 + 5y &= 15 \\ \frac{1}{2}y^{-1/2}dy - 4xdx + 5dy &= 0 \\ \left(\frac{1}{2}y^{-1/2} + 5\right)\frac{dy}{dx} - 4x &= 0 \qquad (\div \; by \; dx) \\ \frac{dy}{dx} &= \frac{4x}{\left(\frac{1}{2}y^{-1/2} + 5\right)} \end{aligned}$$

When you are using implicit differentiation, there are two things to remember:

- First: All the rules apply as before
- Second: you are ASSUMING that you can rewrite the equation in the form $y = f(x)$

Example: Special application of the product rule.
Suppose you want to implicitly differentiate

$$xy = 24$$

what do we do here?

In this case we treat x and y as seperate functions and apply the product rule

$$\begin{aligned} x\frac{dy}{dx} + y\frac{dx}{dx} &= 0 \\ x\frac{dy}{dx} + y &= 0 \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

Alternatively, we could first solve for y, then take the derivative

$$\begin{aligned} xy &= 24 \\ y &= \frac{24}{x} = 24x^{-1} \\ \frac{dy}{dx} &= (-1)24x^{-2} = -\frac{24}{x^2} \end{aligned}$$

which does not look like what we got with implicit differentiation, but, if we use a substitution trick, remembering that originally $xy = 24$, we will get

$$\begin{aligned} \frac{dy}{dx} &= -\frac{24}{x^2} = -\frac{xy}{x^2} \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

Lets try it again

$$\begin{aligned} 48 &= x^2y^3 \\ 0 &= 3x^2y^2\frac{dy}{dx} + 2xy^3\frac{dx}{dx} \qquad \text{(Product rule and power-function rule)} \\ 3x^2y^2\frac{dy}{dx} &= -2xy^3 \qquad \left(\text{again } \frac{dx}{dx} = 1\right) \\ \frac{dy}{dx} &= -\frac{2xy^3}{3x^2y^2} = -\frac{2y}{3x} \end{aligned}$$

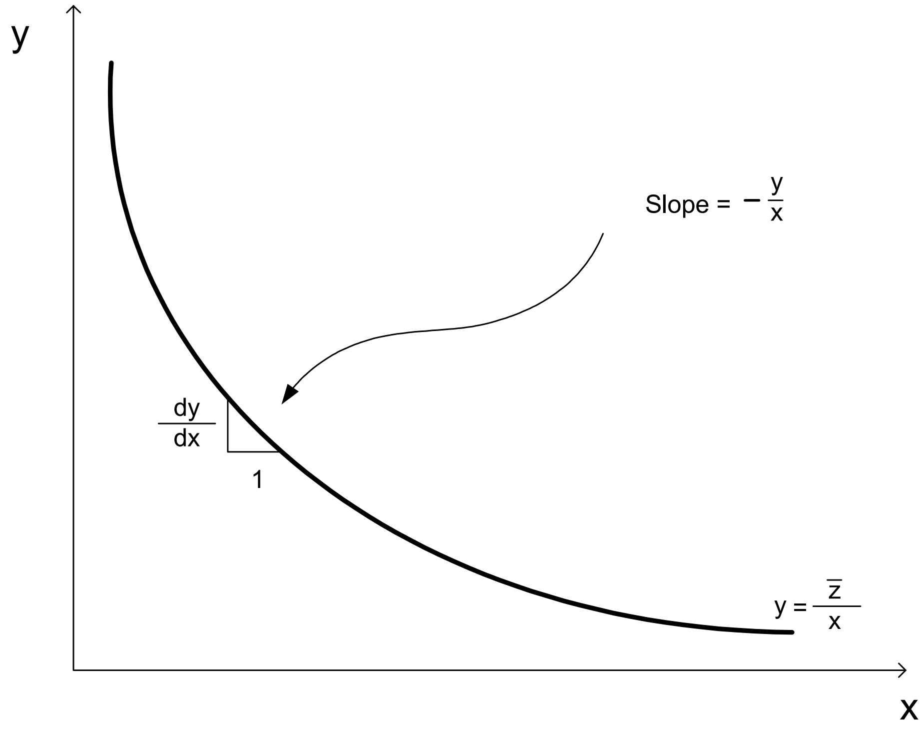

Figure 1:

Level Curves

If we have a function like $z = xy$ or $u = \ln x + \ln y$, then z and u are both functions of x and y. IF we fix z and u to be some particular values such as

$$z = \bar{z} \quad and \quad u = \bar{u}$$

then $\bar{z}$ and $\bar{u}$ are now treated as constants and we are evaluating the functions $\bar{z} = xy$ and $\bar{u} = \ln x + \ln y$ at a particular level. In other words, we are looking for values of x and y that keep z or u constant. This allows us to assume that y is an implicit function of x, i.e.

$$\begin{aligned} yx &= \bar{z} \\ y &= \frac{\bar{z}}{x} \end{aligned}$$

using implicit differentiation, we can find the slope of the level curve

$$\begin{aligned} yx &= \bar{z} \\ x\frac{dy}{dx} + y\frac{dx}{dx} &= \frac{d(\bar{z})}{dx} = 0 \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

The level curve is illustrated in figure 1

In figure 1 we have graphed y as a function of x and a constant, $\bar{z}$. This curve plots all combinations of x and y that keep z at a constant level. Common examples of level curves in economics are *"indifference curves"* (constant utility) and *"isoquants"* (constant levels of output).

Lets look at the utility function example

$$u = \ln x + \ln y$$

where $u = \bar{u}$. using implicit differentiation and the rule of logarithm derivatives

$$\begin{aligned}
\frac{d(\bar{u})}{dx} &= \left(\frac{1}{x}\right) + \left(\frac{1}{y}\right)\frac{dy}{dx} = 0 \\
\frac{dy}{dx} &= -\frac{\frac{1}{x}}{\frac{1}{y}} = -\frac{y}{x}
\end{aligned}$$

Alternatively, we could try to first write this function such that we explicitly have y as a function of x. However, this would require us to "unlog" the function, i.e.

$$\begin{aligned}
\bar{u} &= \ln x + \ln y \\
\bar{u} &= \ln(xy) \\
e^{\bar{u}} &= xy \qquad \text{(unlogged)} \\
y &= \frac{e^{\bar{u}}}{x}
\end{aligned}$$

The result does not look easier to work with than when we used implicit differentiation. This is an example of where implicit differentiation would be preferred.