OPMT 7701

Working with cost functions and production functions

Costs:

Given the total cost function

$$C = Q^3 - 18Q^2 + 120Q$$

the marginal cost is found by taking the derivative of C

$$\frac{dC}{dQ} = MC = 3Q^2 - 36Q + 120$$

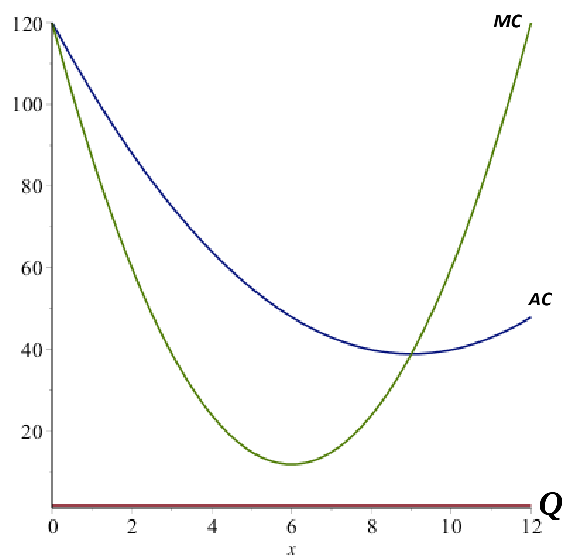
note that the total cost is a "cubic" function and the marginal cost is a "quadratic" function.

The average cost function is found by dividing costs by Q

$$\frac{C}{Q} = AC = Q^2 - 18Q + 120$$

note that average cost (AC) is also a "quadratic" equation.

IMPORTANT: Marginal Cost intersects Average Cost at the MINIMUM of Average Cost. See Figure 1



To test this statement:

First, minimize average cost (AC). Take the derivative of AC and set it to zero. Then find Q

$$\begin{aligned} \frac{dAC}{dQ} &= 2Q - 18 = 0 \\ Q &= 9 \end{aligned}$$

verify that this is a minimum

$$\frac{d^2AC}{dQ^2} = AC'' = 2 > 0$$

Substitute $Q = 9$ into AC

$$\begin{aligned} AC &= Q^2 - 18Q + 120 \\ &= (9)^2 - 18(9) + 120 \\ &= 39 \end{aligned}$$

Now subsititute $Q = 9$ into MC

$$\begin{aligned} MC &= 3Q^2 - 36Q + 120 \\ &= 3(9)^2 - 36(9) + 120 \\ &= 39 \end{aligned}$$

we see that $MC = AC$ at $Q = 9$, which is also the value of Q that minimizes AC

Now find the mimum of marginal cost (MC)

$$\begin{aligned} \frac{dMC}{dQ} &= MC' = 6Q - 36 = 0 \\ Q &= 6 \end{aligned}$$

check the second derivative to verify a minimum

$$\frac{d^2MC}{dQ^2} = MC'' = 6 > 0$$

Production Functions (cubic)

Suppose output, Q, is a function of one input, labour (L) and is given by

$$Q = 12L^2 - L^3$$

Note that output, Q, is also known as TOTAL PRODUCT (TP).

$$TP = 12L^2 - L^3$$

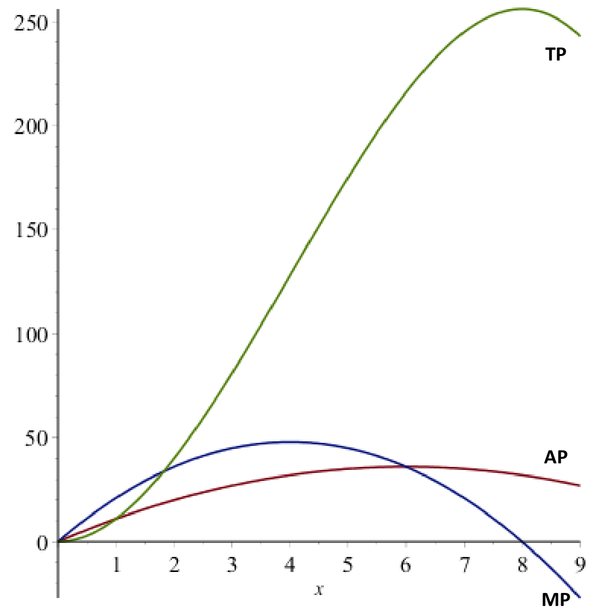
To find the marginal product (MP), take the derivative of total product

$$MP = TP' = 24L - 3L^2$$

and to find average product, divide total product by L

$$\frac{TP}{L} = AP = 12L - L^2$$

The three functions are illustrated in figure 2:



Each of these functions have a maximum. Also, when average product is at a maximum, it is equal to marginal product. When total product is at a maximum, marginal product equals zero

1. Maximize Total product

$$\begin{aligned} TP &= 12L^2 - L^3 \\ TP' &= 24L - 3L^2 = 0 \end{aligned}$$

   there are two solutions, found by factoring:

$$\begin{aligned} L(24 - 3L) &= 0 \\ L &= 0, 8 \end{aligned}$$

2. Maximize Average Product

$$\begin{aligned} AP &= 12L - L^2 \\ AP' &= 12 - 2L = 0 \\ L &= 6 \\ & \quad \textit{and} \\ AP'' &= -2 < 0 \end{aligned}$$

   second derivative is negative, so a maximum

3. Maximize Marginal Product

$$\begin{aligned} MP &= 24L - 3L^2 \\ MP' &= 24 - 6L = 0 \\ L &= 4 \\ & \quad \textit{and} \\ MP'' &= -6 < 0 \end{aligned}$$

   second derivative is negative, so a maximum

4. Check that $MP = AP$ when AP is at a maximum.

$$\begin{aligned} AP &= 12L - L^2 \\ &= 12(6) - 6^2 = 36 \end{aligned}$$

   and marginal product

$$\begin{aligned} MP &= 24L - 3L^2 \\ &= 24(6) - 3(6)^2 = 36 \end{aligned}$$

For doing applications of one variable maximums and minimums, you will often be asked questions involving either production or costs. It is important that you understand:

1. The relationship between total cost (C), marginal cost (MC) and average cost (AC). You should be able to derive MC and AC from the cost function. You should be able to minimize AC and verify that $MC = AC$ at the minimum of AC.

2. The relationship between the production function, marginal product (MP) and average product (AP). It is important to remember that output (Q) and total product (TP) are the same thing.